PMM U.S.S.R.,Vol.45,pp.518-529
0021-8928/82/4 0518 $7.50/0

UDC 539.3

ON THE ENERGY OF AN ELASTIC ROD*

V.L. BERDICHEVSKII

It is shown that the general geometrically nonlinear problem of the three-dimensional theory of elasticity for a rod can be split into a nonlinear "one-dimensional" problem (one-dimensional theory of rods) and a linear "two-dimensional" problem. A number of constant "effective" elastic characteristics of the rod is in the equation of the one-dimensional theory, and they are determined by having the linear two-dimensional problem solved (the section problem). The section problem is formulated in the general case of inhomogeneity and anisotropy as a problem on the minimum of a certain functional (the results presented in /1/, wherein reasoning from /2/ is used, are elucidated in this part). The properties of the section problem are formulated. A double variational problem is constructed. Certain two-sided estimates are given for the effective elastic moduli of inhomogeneous rods. A criterion is obtained for the validity of the strength of materials formula to evaluate the effective tension and bending moduli of an inhomogeneous rod.

1. One-dimensional theory of rods. A curve Γ provided with the ortho triad τ_a $(a, b, c = 1, 2, 3)$ one of whose vectors, τ_3 for definiteness, is tangent to Γ, is modeled in the classical theory of rods. For a fixed position of the curve Γ, the ortho triad is determined to the accuracy of a rotation around the tangent vector. The appropriate degree of freedom describes the rotation of the cross-sections.

The deformed state of the rod is given by the components $r^i(\xi)$, the radius-vector of points of the curve Γ and the components $\tau_a^{\ i}(\xi)$ of the vectors τ_a (ξ is a parameter on Γ, the subscripts i, j, k correspond to the projections on the axes of the Cartesian coordinate system of the observer and run through the values 1,2,3; the quantities with the super and subscripts agree; the site of the index is selected in conformity with the rule of summation over repeated sub- and superscripts).

The quantities $\tau_a^{\ i}$ satisfy the constraints

$$\tau_a^i \tau_{ib} = \delta_{ab}, \quad dr^i/ds = \tau_3^i \tag{1.1}$$

where s is the arclength along Γ, and δ_{ab} is the Kronecker delta.

The deformed state of the rod has four functionally independent degrees of freedom.

In the unstrained state the curve Γ occupies the position Γ_0, determined by a radius-vector with the components $r_{(0)}^i(\xi)$; in the initial state the ortho triad vector components are denoted by $\tau_{(0)a}^{\ \ \ i}(\xi)$. It is understood that the vectors $\tau_{(0)1}$ and $\tau_{(0)2}$ are related to the geometry of the cross-section (for instance, are directed along the axes of inertia of the cross-section or to the physical properties of the rod. We consider the parameter ξ to be the arclength on Γ_0, $0 \leqslant \xi \leqslant |\Gamma_0|$, where $|\Gamma_0|$ is the length of Γ_0. The extension of the rod is characterized by the quantity $\gamma = {}^1/_2 (dr^i/d\xi \cdot dr_i/d\xi - 1)$.

Let ω^c and $\omega_0^{\ c}$ denote the quantities

$$\omega^c = {}^1/_2\, e^{abc} \tau_{ib} d\tau_a^i/ds, \quad \omega_{(0)}^c = {}^1/_2\, e^{abc} \tau_{(0)ib} d\tau_{(0)a}^{\ \ \ i}/d\xi \tag{1.2}$$

where e^{abc} is the Levi-Civita symbol.

It follows from the first equality in (1.1) and from (1.2) that the following relationships (the analog of the Frenet formulas) are valid

$$d\tau_a^i/ds = e_{abc}\omega^c\tau^{ib}, \quad d\tau_{(0)a}^{\ \ \ i}/d\xi = e_{abc}\omega_{(0)}^c\tau_{(0)}^{ib} \tag{1.3}$$

Let us define the measure of rod strain $\overline{\Omega}$ by the equalities

*Prikl.Matem.Mekhan.,45,No.4,704-718,1981

$$\bar{\Omega}^c = (1+2\gamma)^{1/2}\omega^c - \omega^c_{(0)} = {}^1/_2\, e^{abc}(\tau_{ib} d\tau^i_a/d\xi - \tau_{(0)ib} d\tau_{(0)a}^{\ i}\, d\xi) \tag{1.4}$$

We agree that the small Greek letters run through the values 1,2, and the index 3 will be omitted when confusion will not ensue (for instance $\tau^i_3 \equiv \tau^i$, $\tau_{(0)3}^{\ i} \equiv \tau^i_{(0)}$, $\omega^3 \equiv \omega$, $\omega^3_{(0)} \equiv \omega_{(0)}$, $\bar{\Omega}^3 \equiv \bar{\Omega}$).

The quantities $\bar{\Omega}^\alpha$ and $\bar{\Omega}$ are used in classical theory as measures of the bending and torsion of the rod, respectively.

To simplify the tensor mode of writing the further relationships, we take the quantities $\Omega_\beta = e_{\alpha\beta}\bar{\Omega}^\alpha$ as the measure of the bending in place of $\bar{\Omega}^\alpha$, where $e_{\alpha\beta}$ is the two-dimensional Levi-Civita symbol ($e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$). The upper bar is omitted over the measure of the torsion $(\Omega \equiv \bar{\Omega})$.

The functional of the rod energy on which the dead external forces act in classical theory has the form

$$I = \int_0^{|\Gamma_0|} \Phi(\gamma, \Omega_\alpha, \Omega)\, d\xi - L \tag{1.5}$$

where Φ is the internal energy per unit length of rod, and L is the work of the external forces.

The expression (1.5) will be derived below as a result of an asymptotic analysis of the energy functional of a three-dimensional elastic body by the method elucidated in /1,2/. The main result is a formula to evaluate the energy density Φ by means of physical characteristics and the geometry of the rod cross-section. The energy density Φ of a physically linear anisotropic inhomogeneous rod is given by the equality

$$\Phi = {}^1/_2 \langle E(\gamma + \xi^\alpha \Omega_\alpha)^2 \rangle + \Psi(\gamma, \Omega_\alpha, \Omega) \tag{1.6}$$

where the angular brackets are the integral over the domain of the cross-section S, ξ^α are Cartesian coordinates in the cross-section, E is Young's modulus, $\Psi(\gamma, \Omega_\alpha, \Omega)$ is the minimal value of a certain functional Θ which is a quadratic functional of the three functions y_α and y defined in the domain S that depends on γ, Ω_α and Ω as on the parameters

$$\Psi(\gamma, \Omega_\alpha, \Omega) = \inf_{y_\alpha, y} \Theta(y_\alpha, y; \gamma, \Omega_\alpha, \Omega) \tag{1.7}$$

The functional Θ is the sum of two positive quadratic functionals $\Theta_\angle$ and $\Theta_\perp$ of the form

$$\Theta_\angle = {}^1/_2 \langle G^{\alpha\beta}(y_{|\alpha} + h\Omega e_{\sigma\alpha}\zeta^\sigma + C_\alpha(\gamma + h\Omega_\sigma\zeta^\sigma))(\alpha \to \beta)\rangle \tag{1.8}$$

$$\Theta_\perp = {}^1/_2 \langle C \qquad (y_{(\alpha|\beta)} + C_{\alpha\beta}(\gamma + h\Omega_\sigma\zeta^\sigma) + C^\lambda_{\alpha\beta}(y_{|\lambda} + h\Omega e_{\sigma\lambda}\zeta^\sigma))(\alpha, \beta \to \gamma, \delta)\rangle \tag{1.9}$$

Here $G^{\alpha\beta}$, C_α, $C^{\alpha\beta\gamma\delta}$, $C_{\alpha\beta}$, $C^\lambda_{\alpha\beta}$ are "two-dimensional projections" of the elastic moduli tensor, h is the diameter of the cross-section, $\zeta^\alpha = \xi^\alpha/h$ the vertical bar separating the Greek subscripts denotes differentiation with respect to ζ^α, the symbol $(\alpha \to \beta)$ denotes the preceding brackets with the subscript α replaced by β. The function $y(\zeta^\alpha)$ has the meaning of a dimensionless warping of the cross-section, while the function $y_\alpha(\zeta^\beta)$ is the displacement component in the plane of the cross-section. The functional $\Theta_\angle$ is independent of y_α.

In the particular case of an isotropic homogeneous rod, as is shown below, $\inf \Theta_\perp = 0$ and the problem of the minimum of the functional $\Theta_\angle$ is equivalent to the torsion problem of Saint-Venant. The function Ψ is given by the equality $\Psi = {}^1/_2 C\Omega^2$, where C is the torsional stiffness of the rod; (1.6) here goes over into the known formula from strength of materials. For anisotropic or inhomogeneous rods, the function Ψ contains additional components whose evaluation is related to the solution of the variational problem (1.7).

2. A three-dimensional functional. Let us consider an elastic, inhomogeneous anisotropic body occupying a domain V_0 in its undeformed state. The domain V_0 is formed by motion along the space curve Γ_0 (the rod axis) of a plane figure S at each point of the perpendicular axis. Let us introduce a curvilinear coordinate system ξ^α, ξ in V_0 by means of the formulas

$$x^i = r_{(0)}^{\ i}(\xi) + \tau_{(0)\alpha}^{\ i}\xi^\alpha \tag{2.1}$$

It is assumed that $\xi^\alpha = 0$ is the center of gravity of S, i.e., $\langle \xi^\alpha \rangle = 0$. The coordinates ξ^α, ξ are considered co-moving.

Values of the metric tensor components $g_{(0)\alpha\beta}$, $g_{(0)\alpha 3}$, $g_{(0)33}$ in the coordinate system ξ^α, ξ are presented in /3/, for example.

Let us consider that $\omega^a_{(0)}$ is a continuously differentiable function of ξ. For sufficiently small numbers R the following inequalities are valid

$$|\omega_{(0)}^{\alpha}| \leqslant R^{-1}, \quad |d\omega_{(0)}^{\alpha}/d\xi| \leqslant R^{-2} \tag{2.2}$$

We call the best constant in the inequalities (2.2) (the maximum of the numbers R for which (2.2) hold), the characteristic scale of the curvature and torsion. It is assumed that $h \ll R$.

Let us consider the problem of the equilibrium of an elastic rod subjected to dead surface and volume forces under the condition that the following particle positions are given on the rod endfaces

$$x^i(\xi^\alpha, 0) = a^i + b_\alpha^i \xi^\alpha, \quad x^i(\xi^\alpha, |\Gamma_0|) = a_1^i + b_{1\alpha}^i \xi^\alpha \tag{2.3}$$

Here $x^i(\xi^\alpha, \xi)$ are functions governing the particle position in the deformed state, and a^i, a_1^i, b_α^i, $b_{1\alpha}^i$ are given constants.

Rod equilibria correspond to stationary points of the functional

$$I = \int_{\Gamma_*} d\xi \left(\langle \varkappa U \rangle - \langle F_i x^i(\xi^\alpha, \xi) \rangle - \int_{\partial S} P_i x^i(\xi^\alpha, \xi)\, d\sigma \right) \tag{2.4}$$

in the set of functions $x^i(\xi^\alpha, \xi)$ which satisfy the constraints (2.3). The U in (2.4) is a given function of the strain tensor components, $2\varepsilon_{\alpha\beta} = x_{,\alpha}^i x_{i,\beta} - g_{(0)\alpha\beta}$, $2\varepsilon_{\alpha 3} = x_{,\alpha}^i x_{i,\xi} - g_{(0)\alpha 3}$, $2\varepsilon_{33} = x_{,\xi}^i x_{i,\xi} - g_{(0)33}$, the comma in front of the ξ in the subscripts denotes differentiation with respect to ξ, the comma in front of the Greek subscripts denotes differentiation with respect to ξ^α, $\varkappa$ is a determinant, the initial metric $\varkappa = 1 + \omega_{(0)\alpha}\xi^\alpha$, F_i and P_i are the volume and surface force components.

For small h it is required to replace the problem of finding the stationary points of the functional (2.4) by an approximate "one-dimensional" problem in which only functions of ξ figure.

We shall limit ourselves to the examination of small strains and we start with an investigation of a physically linear elastic body, when U is a quadratic form in the strains.

Let us represent U in the form of the sum of three positive-definite quadratic forms

$$\begin{aligned} U &= U_{\parallel} + U_{\angle} + U_{\perp} \\ U_{\parallel} &= {}^1\!/_2 E_{\parallel} \varepsilon_{33}^2, \quad U_{\angle} = {}^1\!/_2 G_{\angle}^{\alpha\beta} (2\varepsilon_{\alpha 3} + E_\alpha \varepsilon_{33})(\alpha \to \beta) \\ U_{\perp} &= {}^1\!/_2 E^{\alpha\beta\gamma\delta} (\varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E_{\alpha\beta}^{\sigma} 2\varepsilon_{\sigma 3})(\alpha, \beta \to \gamma, \delta) \end{aligned} \tag{2.5}$$

From the formula

$$U_{\parallel} = \min_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha 3}} U, \quad U_{\angle} = \min_{\varepsilon_{\alpha\beta}} (U - U_{\parallel}), \quad U_{\perp} = U - U_{\parallel} - U_{\angle}$$

it follows that this representation is unique, and the two dimensional tensors $E^{\alpha\beta\gamma\delta}$, $G_{\angle}^{\alpha\beta}$, $E_{\alpha\beta}$, $E_{\alpha\beta}^{\delta}$, E_α, $E_{\parallel}$ can be considered as independent components of the three-dimensional elastic moduli tensor. This latter can be expressed in terms of two-dimensional tensors if the brackets in (2.5) are expanded.

We shall construct a "one-dimensional" functional by using the passage to the limit $h \to 0$. The papers /4 – 8/ are devoted to the appropriate asymptotic analysis of the elasticity theory equations. Formulation of the problem should be supplemented by an indication of the dependence of the characteristics of the external actions P_i and F_i and of the elastic moduli tensor components on h. Something will be said about this below.

We make the change of variables $\xi^\alpha = h\zeta^\alpha$. After replacement of the domain of the change of variables ζ^α, ξ is independent of h, and the parameter h enters the functional explicitly. The domain of variation of the variables ζ^α exactly as the domain of variation of ξ^α, is denoted by S.

3. Replacement of the sought functions. We introduce the functions

$$r^i(\xi) = \langle x^i(\xi^\alpha, \xi) \rangle / |S| \tag{3.1}$$

where $|S|$ is the area of the cross-section S.

We call the curve $x^i = r^i(\xi)$ the deformed axis of the rod.

We endow the curve Γ with two unit vectors $\tau_\alpha^i(\xi)$, which are mutually orthogonal and orthogonal to the vector $\tau^i = dr^i/ds$.

We make the change in the sought functions $x^i(\xi^\alpha, \xi) \to y^i(\zeta^\alpha, \xi)$

$$x^i(\xi^\alpha, \xi) = r^i(\xi) + h\tau_\alpha^i(\xi)\zeta^\alpha + hy^i(\zeta^\alpha, \xi) \tag{3.2}$$

Because of the definition of $r^i(\xi)$ (3.1), the functions y^i satisfy the constraints

$$\langle y^i\rangle = 0 \tag{3.3}$$

The additional degree of freedom that occurs with the assignment of τ_α^i permits imposition of still another constraint on y^i. For definiteness, we take the equality

$$e^{\alpha\beta}\langle y_{\alpha|\beta}\rangle = 0, \quad y_\alpha \equiv \tau_\alpha^i y_i \tag{3.4}$$

as such a constraint.

The equality (3.2) sets up a mutually one-to-one correspondence between all the functions $x^i(\xi^\alpha, \xi)$ and all the sets $\{r^i(\xi), \tau_\alpha{}^i(\xi), y^i(\zeta^\alpha, \xi)\}$ subject to the constraints (3.3) and (3.4), and the orthonormality condition of the triad τ_a (1.1).

The functions r^i and $\tau_\alpha{}^i$ at the rod endfaces are selected in conformity with the boundary conditions (2.1)

$$r^i(0) = a^i, \quad r^i(|\Gamma_0|) = a_1{}^i, \quad \tau_\alpha^i(0) = b_\alpha^i, \quad \tau_\alpha^i(|\Gamma_0|) = b_{1\alpha}^i \tag{3.5}$$

Here

$$y^i(\zeta^\alpha, 0) = y^i(\zeta^\alpha, |\Gamma_0|) = 0 \tag{3.6}$$

4. Characteristic scale.

We define the amplitude of the bending-torsion deformation ε_Ω and the amplitude of the extension of axis ε_γ by the formulas

$$\varepsilon_\Omega = h\max\nolimits_\xi(\Omega_a\Omega^a)^{1/2}, \quad \varepsilon_\gamma = \max\nolimits_\xi|\gamma|$$

The quantity $\varepsilon = \varepsilon_\Omega + \varepsilon_\gamma$ characterizes the amplitude of the deformations.

We introduce the characteristic scale of variation of the state of deformation l as the best constant in the system of inequalities

$$|\Omega_{a,\xi}| \leqslant \varepsilon_\Omega l^{-1}, \quad |\gamma_{,\xi}| \leqslant \varepsilon_\gamma l^{-1}, \quad |y_{i,\xi}| \leqslant l^{-1}\max_S(y^i_{|\alpha}y_i^{|\alpha})^{1/2} \tag{4.1}$$

The scale l is a function of ξ. We assume that the scale l is commensurate with h in the neighborhood of the endfaces $0 \leqslant \xi \leqslant b$, $|\Gamma_0| - b \leqslant \xi \leqslant |\Gamma_0|$, $b \sim h$, while $h_{**} = h/l \ll 1$ far from the endfaces.

5. Elastic moduli and external forces.

The components of the tensors $E^{\alpha\beta\gamma\delta}$ $G_\angle^{\alpha\beta}$, $E_\parallel$ have the dimensionality of the shear modulus G, while the tensor components $E_{\alpha\beta}{}^\gamma$ $E_{\alpha\beta}$, E_α are dimensionless. Because of the positive-definiteness of $U_\parallel$, $U_\angle$ and $U_\perp$ the tensors $E^{\alpha\beta\gamma\delta}$ and $G_\angle^{\alpha\beta}$ are positive definite, while $E_\parallel$ is a positive scalar. The dimensionless tensors $E_{\alpha\beta}{}^\gamma$, $E_{\alpha\beta}$ and E_α can take on arbitrary values.

We assume that as $h \to 0$

$$\begin{gathered} E_\parallel = E(\zeta^\alpha, \xi) + O(Gh_*), \quad G_\angle^{\alpha\beta} = G^{\alpha\beta}(\zeta^\alpha, \xi) + O(Gh_*) \\ E^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta}(\zeta^\alpha, \xi) + O(Gh_*), \quad E_{\alpha\beta}^\sigma = C_{\alpha\beta}^\sigma(\zeta^\alpha, \xi) + O(Gh_*) \\ E_{\alpha\beta} = C_{\alpha\beta}(\zeta^\alpha, \xi) + O(Gh_*), \quad E_\alpha = C_\alpha(\zeta^\alpha, \xi) + O(Gh_*), \quad h_* \equiv h/R \end{gathered} \tag{5.1}$$

If the elastic properties are symmetric relative to a plane perpendicular to the rod axis, then the two-dimensional tensors with an odd number of indices vanish: $C_{\alpha\beta}{}^\gamma = C_\alpha = 0$. If, in addition, the elastic properties are invariant relative to rotation in the cross-sectional plane (a transversely isotropic body), then in conformity with the general theory of tensor functions /9/:

$$C^{\alpha\beta\gamma\sigma} = \lambda\delta^{\alpha\beta}\delta^{\gamma\sigma} + \mu(\delta^{\alpha\gamma}\delta^{\beta\sigma} + \delta^{\alpha\sigma}\delta^{\beta\gamma}), \quad G^{\alpha\beta} = G\delta^{\alpha\beta}, \quad C^{\alpha\beta} = \nu\delta^{\alpha\beta}$$

The elastic properties of such a body are determined by the parameters E, G, λ, μ, ν, where E, G, μ and $\lambda + \mu$ are positive, while the quantity ν is arbitrary. For an isotropic body $E = 2\mu(1 + \nu)$ is Young's modulus, $G = \mu$ is the shear modulus, and $\nu = {}^1/_2\lambda(\lambda + \mu)^{-1}$ is the Poisson's ratio. In the anisotropic case E is naturally called the longitudinal Young's modulus, $G^{\alpha\beta}$ is the tensor of the shear modulus, and $C^{\alpha\beta}$ is the tensor of the (transverse) Poisson's ratios.

Let us turn attention to the fact that the "prelimit" two-dimensional tensors of odd rank differ from zero because the coordinate system, is curvilinear, even for an isotropic body, as is seen from their expressions presented in /3/.

For the external forces we take the conditions

$$P_i = O(G\varepsilon h_{**}), \quad F_i = O(G\varepsilon l^{-1})$$

and for simplicity, we limit ourselves to the consideration of external volume forces which are constant over the cross-section.

6. Asymptotic analysis of the energy functional. Let us construct a theory of rods in a first approximation. This means that all quantities on the order of ε, h_* and h_{**} are neglected as compared with one.

Let us determine $r^i(\xi)$ and $\tau_\alpha{}^i(\xi)$ and we seek y^i in a first approximation. We assume $y^i \sim \varepsilon$. Then, in a first approximation, the equalities

$$\varepsilon_{\alpha\beta} = y_{(\alpha|\beta)}, \quad 2\varepsilon_{\alpha 3} = y_{|\alpha} + h\Omega e_{\sigma\alpha}\zeta^\sigma, \quad \varepsilon_{33} = \gamma + h\Omega_\sigma\zeta^\sigma \tag{6.1}$$

are valid for the strain components.

Here $y = (1+2\gamma)^i \tau^i y_i$, and symmetrization is noted by the parentheses in the subscripts. The derivatives of y^i with respect to ξ do not enter into the functional, it does not "maintain" the boundary conditions (3.6), and seeking y and y_α reduces to minimizing the quantity $\Theta = \Theta_\angle + \Theta_\perp$, for each ξ, while $\Theta_\angle$ and $\Theta_\perp$ are given by (1.8) and (1.9).

The minimum is sought over all functions y_α and y satisfying the constraints

$$\langle y\rangle = 0, \quad \langle y_\alpha\rangle = 0, \quad e^{\alpha\beta}\langle y_{\alpha|\beta}\rangle = 0 \tag{6.2}$$

in conformity with (3.3) and (3.4).

The work of the external forces is discarded because of the estimates (5.1). The minimizing functions of the functional Θ are evidently of order ε. This yields the foundation for the validity of the assumption made.

The minimal value Ψ of the functional Θ is a certain quadratic form of the parameters γ, Ω_α and Ω. According to (2.5), (5.1) and (6.1), the linear energy density of the rod in which just terms on the order of $Gh^2\varepsilon^2$ are retained is given by (1.6).

The first term in (1.6) characterizes the tensile and bending energy, the second the torsional energy and the additional contribution to the tensile and bending energy from the transverse strain.

7. Investigation of the section problem. Homogeneous rods. Finding the minimizing functions of the functional Θ is equivalent to solving the Neumann boundary value problem for a system of three second-order elliptic equations with variable coefficients in y_α, y. It is equivalent to a mixed boundary value problem for a system of second and fourth order equations in y and the stress function χ which is obtained by passing to the dual variational problem by the general rule /10/. We present its formulation

$$\Psi(\gamma, \Omega_\alpha, \Omega) = \inf_y \sup_\chi [\Theta_\angle(y) + \Theta_\perp^*(\chi, y)] \tag{7.1}$$

$$\Theta_\perp^*(\chi, y) = \langle e^{\alpha\mu}e^{\beta\nu}\chi_{|\mu\nu}[C_{\alpha\beta}(\gamma + h\Omega_\sigma\zeta^\sigma) + C_{\alpha\beta}^{\lambda}(y_{|\lambda} + h\Omega e_{\sigma\lambda}\zeta^\sigma)] - {}^1/_2\, C^{-1}_{\alpha\beta\gamma\delta}e^{\alpha\mu}e^{\beta\nu}e^{\gamma\lambda}e^{\delta\varkappa}\chi_{|\mu\nu}\chi_{|\lambda\varkappa}\rangle$$

where the upper bound is sought in all functions χ satisfying the conditions

$$\chi_{|\alpha} = \text{const} \tag{7.2}$$

on the boundary of the domain S, and $C^{-1}_{\alpha\beta\gamma\delta}$ is the inverse tensor to $C^{\alpha\beta\gamma\delta}$ in the sense that $C^{-1}_{\alpha\beta\gamma\delta}C^{\gamma\delta\mu\nu} = \delta^\mu_{(\alpha}\delta^\nu_{\beta)}$.

As a rule, the variational problems formulated can be solved only be numerical methods. There are individual cases, some of which will be examined, when sufficiently essential information is successfully obtained by elementary means.

The results from /11,12/ will be elucidated below in this section in variational terminology and tensor form (see the monograph /13/ also).

Let a rod be homogeneous in the transverse directions, i.e., $C^{\alpha\beta\gamma\delta}$, $G^{\alpha\beta}$, $C_{\alpha\beta}{}^\lambda$, $C_{\alpha\beta}$ and C_α are independent of ζ^α. Then for a rod with arbitrary cross-sectional geometry and anisotropy of general form Ψ is independent of γ, and measures of the bending and torsion enter Ψ in the form of the combinations $\Omega_* = \Omega - {}^1/_2\, e^{\mu\nu}C_\mu\Omega_\nu$, hence

$$\Psi = {}^1/_2\, C\Omega_*{}^2 \tag{7.3}$$

Here the torsional stiffness C is the minimum value of the functional

$$C = \inf_{z,\, z_\alpha} h^2 \langle G^{\alpha\beta}(z_{|\alpha} + e_{\sigma\alpha}\zeta^\sigma)(\alpha \to \beta) + C^{\alpha\beta\gamma\delta}(z_{(\alpha|\beta)} + C^\lambda_{\alpha\beta} z_{|\lambda})(\alpha, \beta \to \gamma, \delta)\rangle \tag{7.4}$$

Indeed, let us replace the requiredfunctions

$$\begin{aligned} y &= -C_\alpha \gamma\zeta^\alpha - {}^1\!/_2 C_\alpha h\Omega_\beta (\zeta^\alpha\zeta^\beta - \langle\zeta^\alpha\zeta^\beta\rangle/|S|) + h\Omega_* z \\ y_\alpha &= -(C_{\alpha\beta} - C^\lambda_{\alpha\beta} C_\lambda)\gamma\zeta^\beta - {}^1\!/_2 a_{\alpha\beta\gamma}(\zeta^\beta\zeta^\gamma - \langle\zeta^\beta\zeta^\gamma\rangle/|S|) + h\Omega_* z_\alpha \end{aligned} \tag{7.5}$$

We take solutions of the linear system of equations

$$a_{(\alpha\beta)\gamma} \equiv {}^1\!/_2 (a_{\alpha\beta\gamma} + a_{\beta\alpha\gamma}) = C_{\alpha\beta} h\Omega_\gamma - C^\sigma_{\alpha\beta} h\Omega_{(\sigma} C_{\gamma)} - C^\sigma_{\alpha\beta} h\Omega c_{\gamma\sigma} \tag{7.6}$$

as the constants $a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta}$

It can be seen by direct substitution that for any tensor of the third rank $a_{\alpha\beta\gamma}$ that is symmetric relative to the last two subscripts, the following identity holds

$$a_{\alpha\beta\gamma} = a_{(\alpha\beta)\gamma} + a_{(\alpha\gamma)\beta} - a_{(\beta\gamma)\alpha} \tag{7.7}$$

This latter equality (7.7) yields the solution of the system (7.6).

Substituting (7.5) into (1.7) and (1.8) with (7.6) taken into account results in the relationships (7.3) and (7.4). There remains still to note that the constraints (3.3) and (3.4) are not essential to the evaluation of the lower boundary of the functional Θ since it is invariant with respect to the substitutions $y \to y + c$, $y_\alpha \to y_\alpha + c_\alpha + \omega e_{\alpha\beta}\zeta^\beta$, c, c_α, ω are constants.

Elliptical rod. Let S be given by the equation $b_{\alpha\beta}\zeta^\alpha\zeta^\beta \leqslant 1$, $b_{\alpha\beta}$ which is a symmetric positive definite tensor. It is not difficult to guess the minimizing functions: they satisfy the equations (a is a constant)

$$G^{\alpha\beta}(z_{|\beta} + e_{\sigma\beta}\zeta^\sigma) = a e^{\alpha\lambda} b_{\lambda\mu}\zeta^\mu, \quad z_{(\alpha|\beta)} + C^\lambda_{\alpha\beta} z_{|\lambda} = 0$$

Hence

$$\begin{aligned} z &= {}^1\!/_2 (a G^{-1}_{\beta\alpha} e^{\alpha\lambda} b_{\lambda\mu} - e_{\mu\beta})(\zeta^\mu\zeta^\beta - \langle\zeta^\mu\zeta^\beta\rangle/|S|) \\ a &= 2(G^{-1}_{\alpha\beta} e^{\alpha\lambda} e^{\mu\beta} b_{\lambda\mu})^{-1}, \\ z_\alpha &= {}^1\!/_2 \bar{a}_{\alpha\beta\gamma}(\zeta^\alpha\zeta^\gamma - \langle\zeta^\beta\zeta^\gamma\rangle/|S|) \end{aligned} \tag{7.8}$$

where $G_{\alpha\beta}^{-1}$ is a tensor inverse to $G^{\alpha\beta}$, $\bar{a}_{\alpha\beta\gamma}$ is the solution of a linear system of equations $\bar{a}_{(\alpha\beta)\gamma} = -C_{\alpha\beta}{}^\lambda (aG_{\sigma\lambda}{}^{-1} e^{\sigma\varkappa} b_{\varkappa\mu} - e_{\mu\lambda})$, which is given by (7.7). Consequently $C = a^2 G^{-1}_{\alpha\beta} e^{\alpha\lambda} e^{\beta\varkappa} b_{\lambda\mu} b_{\varkappa\nu} I^{\mu\nu}$, $I^{\mu\nu} = \langle\xi^\mu\xi^\nu\rangle$ or taking into account that $b_{\alpha\beta} = {}^1\!/_4 |S| h^2 I^{-1}_{\alpha\beta}$ ($I^{-1}_{\alpha\beta} I^{\beta\gamma} = \delta_\alpha{}^\gamma$), for an ellipse, we have

$$C = 4[G^{-1}_{\alpha\beta} e^{\alpha\mu} e^{\beta\nu} I^{-1}_{\mu\nu}]^{-1} \tag{7.9}$$

We turn attention to the fact that the value of the torsional stiffness is obtained without any assumptions about the coaxiality of the tensors $G_{\alpha\beta}$ and $I_{\alpha\beta}$.

Estimate of the torsional stiffness of a homogeneous anisotropic rod of arbitrary cross-section. The estimate

$$C \leqslant 4(G^{-1}_{\alpha\beta} e^{\alpha\mu} e^{\beta\nu} I^{-1}_{\mu\nu}]^{-1} \tag{7.10}$$

is valid for the torsional stiffness.

It is an extension of the inequality /14/ that is written as follows in tensor form

$$C \leqslant 4\mu\,[\,I_\alpha^{-1\alpha}]^{-1} \tag{7.11}$$

to the anisotropic case.

To derive the inequality (7.11) from (7.10), it should be recalled that $G^{-1}_{\alpha\beta} = \mu^{-1}\delta_{\alpha\beta}$ in the anisotropic case.

We substitute (7.8), in which $b_{\alpha\beta}$ are arbitrary parameters, as trial functions into the functional (7.4). This results in the estimate

$$C \leqslant 4\bar{G}^{\mu\nu} b_{\mu\lambda} b_{\nu\varkappa} I^{\lambda\varkappa} [\bar{G}^{\mu\nu} b_{\mu\nu}]^{-2} \tag{7.12}$$

Here we have introduced the temporary notation $\bar{G}^{\mu\nu} = G^{-1}_{\alpha\beta} e^{\alpha\mu} e^{\beta\nu}$.

We now minimize the right side of (7.12) with respect to $b_{\alpha\beta}$. This is equivalent to minimizing the quadratic form $\bar{G}^{\mu\nu} b_{\mu\lambda} b_{\nu\varkappa} l^{\lambda\varkappa}$ under the constraints $\bar{G}^{\mu\nu} b_{\mu\nu} = 1$. It can be shown that the minimum is achieved at $b_{\alpha\beta} = l^{-1}_{\alpha\beta} (\bar{G}^{\lambda\varkappa} l^{-1}_{\lambda\varkappa})^{-1}$ which indeed proves the assertion expressed.

An estimate, analogous (and possibly also equivalent) to (7.10), was constructed in /12/, however it has a very much more complex form.

8. Inhomogeneous rods. In examining inhomogeneous rods, we restrict ourselves to the case when the rod has a plane of elastic symmetry perpendicular to the axis. Here $C_{\alpha\beta}{}^{\gamma} = C_\alpha = 0$ and the problem about the minimum of the functional Θ splits into two independent problems, the problem of the minimum of the functional $\Theta_\angle (y)$, with respect to y, and the problem of the minimum of the functional $\Theta_\perp (y_\alpha)$ with respect to y_α. The former is substantially the Saint-Venant problem on torsion /15,16/. The function minimizing $\Theta_\angle (y)$ is evidently proportional to Ω and $\Psi_\angle = \inf_y \Theta_\angle (y) = {}^1/_2 C\Omega^2$. The latter problem corresponds to a certain plane problem of the linear theory of elasticity which N.I. Muskhelishvili investigated in the isotropic case /17/. The minimizing functions depend linearly on γ and Ω_α hence, the minimal value $\Psi_\perp$ of the functional $\Theta_\perp$ will be a nonnegative quadratic form in γ and Ω_α

$$\Psi_\perp = \inf_{y_\alpha} \Theta_\perp = {}^1/_2 |S| (E_\perp \gamma^2 + 2E_\perp^\alpha \gamma \Omega_\alpha + E_\perp^{\alpha\beta} \Omega_\alpha \Omega_\beta) \tag{8.1}$$

Centrally-symmetric section. The cross-section is said to be centrally symmetric if besides each point with the coordinates ζ^α it contains a point with the coordinates $-\zeta^\alpha$. For functions define in centrally symmetric domains, the concept of evenness can be introduced; each function can be represented in the form of a sum of odd and even functions (they are marked, respectively, by one and two primes).

Let $C^{\alpha\beta\gamma\delta}$ and $C^{\alpha\beta}$ be even functions of ζ^α. Then the functional $\Theta_\perp$ splits into the sum of two functionals

$$\Theta_\perp' = {}^1/_2 \langle C^{\alpha\beta\gamma\delta} (y'_{(\alpha|\beta)} + C_{\alpha\beta}\gamma) (\alpha, \beta \to \gamma, \delta) \rangle$$
$$\Theta_\perp'' = {}^1/_2 \langle C^{\alpha\beta\gamma\delta} (y''_{(\alpha|\beta)} + C_{\alpha\beta} h \Omega_\sigma \zeta^\sigma) (\alpha, \beta \to \gamma, \delta) \rangle$$

They can be minimized independently. The lower bound of $\Theta_\perp'$ is proportional to γ^2, and of $\Theta_\perp''$ to $\Omega_\alpha{}^2$. Thus under the assumptions made $E_\perp^\alpha = 0$. The same deduction is obtained if $C^{\alpha\beta}$ are odd functions of ζ^α.

Inhomogeneous rod with constant Poisson's ratios. It turns out that the following remarkable equality holds for $C_{\alpha\beta} = \text{const}$

$$\Psi_\perp = 0 \tag{8.2}$$

For the proof we use the fact that evaluation of $\Theta_\perp$ by any functions y_α yields the upper bound of $\Psi_\perp$. We put

$$y_\alpha = -C_{\alpha\beta}\gamma\zeta^\beta - {}^1/_2 a_{\alpha\beta\gamma} (\zeta^\beta\zeta^\gamma - \langle \zeta^\beta\zeta^\gamma \rangle / |S|)$$

where $a_{\alpha\beta\gamma}$ is the solution of the system $a_{(\alpha\beta)\gamma} = C_{\alpha\beta} h\Omega_\gamma$. In these functions $\Theta_\perp = 0$. Therefore, $\Psi_\perp \leqslant 0$. Hence (8.2) follows by virtue of the non-negativity of $\Psi_\perp$.

The minimizing functions y_α of the functional $\Theta_\perp$ are universal in form for an arbitrary cross-section and arbitrary dependence on the coordinates of the elastic moduli $C^{\alpha\beta\gamma\delta}$; it is found by using (7.7)

$$y_\alpha = -C_{\alpha\beta}\gamma\zeta^\beta - (C_{\alpha\beta} h\Omega_\gamma - {}^1/_2 C_{\beta\gamma} h\Omega_\alpha) (\zeta^\beta\zeta^\gamma - \langle \zeta^\beta\zeta^\gamma \rangle / |S|)$$

Criterion that $\Psi_\perp$ vanishes. The quadratic form $\Psi_\perp$ equals zero identically not only for rods with constant Poisson's ratios, but also for certain rods with variable Poisson's ratios.

The following assertion is correct.

Let the domain S be divided into two parts S_1 and S_2 by a differentiating line L. The Poisson's ratios in each part are continuous, but become discontinuous on the line L. Then for the quadratic form $\Psi_\perp$ to vanish, it is necessary and sufficient that there exists a function $c(\zeta^\alpha)$ such that in the domain of continuity of $C_{\alpha\beta}$

$$C_{\alpha\beta} = c_{|\alpha\beta} \tag{8.3}$$

and on the line of discontinuity

$$[c_{|\alpha}] = \text{const} \tag{8.4}$$

The symbol $[\cdot]$ denotes the difference between values of the function on both sides of the discontinuity.

Proof. Sufficiency. We put

$$y_\alpha = -c_{|\alpha}(\gamma + h\Omega_\sigma\zeta^\sigma) + ch\Omega_\alpha + a_\alpha + \omega e_{\alpha\sigma}\zeta^\sigma$$

where a_α, ω are constants whose values are different in S_1 and S_2. We select these constants so that the condition $[y_\alpha]=0$ would be satisfied on L. Relying on the condition of kinematic compatibility $d[c]/d\sigma = r_\alpha d\zeta^\alpha/d\sigma$ $(r_\alpha \equiv [c_{|\alpha}]$, σ is a parameter on L from which it follows that $[c] = r_\alpha\zeta^\alpha + r$, $r = \text{const}$, we find that $[y_\alpha]=0$ for $[a_\alpha] = r_\alpha\gamma + rh\Omega_\alpha$, $[\omega] = e^{\sigma\alpha}r_\sigma h\Omega_\alpha$. The functions y_α constructed are allowable and $\Theta_\perp = 0$ on them, therefore $\Psi_\perp = 0$.

Necessity. Let $\Psi_\perp = 0$. Then by virtue of the nonnegativity of $U_\perp U_\perp \equiv 0$ and the plane "deformations" $C_{\alpha\beta}$ and $C_{\alpha\beta}\zeta^\sigma$ should be compatible. It follows from the condition of compatibility in the plane problem that $C_{\alpha\beta}$ should satisfy the system of equations

$$\Delta C_{\alpha\beta} + C^\lambda_{\lambda|\alpha\beta} - C^\mu_{\alpha|\beta\mu} - C^\mu_{\beta|\alpha\mu} = 0 \tag{8.5}$$

$$2C_{\alpha\beta|\lambda} + (C^\nu_{\nu|\alpha} - C^\nu_{\alpha|\nu})\delta_{\lambda\beta} + (C^\nu_{\nu|\beta} - C^\nu_{\beta|\nu})\delta_{\lambda\alpha} - C_{\alpha\lambda|\beta} - C_{\beta\lambda|\alpha} = 0 \tag{8.6}$$

Removing (8.6) in α, β, we obtain $C^\nu_{\nu|\lambda} = C^\nu_{\lambda|\nu}$. Hence, and from (8.6)

$$2C_{\alpha\beta|\lambda} = C_{\alpha\lambda|\beta} + C_{\beta\lambda|\alpha} \tag{8.7}$$

We express $C_{\alpha\lambda|\beta}$ in (8.7) by the formula obtained from (8.7) by replacing the subscripts $\beta \rightleftarrows \lambda$. Taking account of the symmetry of $C_{\alpha\beta}$ we obtain $C_{\beta\alpha|\lambda} = C_{\beta\lambda|\alpha}$. Therefore, there exists a vector $c_\alpha(\zeta^\beta)$ such that $C_{\alpha\beta} = c_{\alpha|\beta}$. The condition of symmetry of $C_{\alpha\beta}$ means that the vector c_α is a potential one, $c_\alpha = c_{|\alpha}$, and therefore, the equalities (8.3) hold. Hence (8.5) and (8.6) are satisfied. The formulas written down above for y_α result from (8.3) and the condition $U_\perp = 0$ and it can be seen that the requirement of continuity of y_α results in the conditins (8.4).

Let us present a number of elementary corollaries of the criterion formulated.

Corollary. 1°. If $\Psi_\perp = 0$ for an inhomogeneous anisotropic rod with discontinuous Poisson's ratios, then on the line of discontinuity these Poisson's ratios satisfy the condition

$$[C_{\alpha\beta}]\,\tau^\beta = 0 \tag{8.8}$$

where τ^β is the tangent vector to the line of discontinuity.

2°. For a transversely isotropic rod $\Psi_\perp \equiv 0$ if an only if the Poisson's ratio ν is constant.

3°. If for an inhomogeneous anisotropic rod with piecewise-constant Poisson's ratios $\Psi_\perp \equiv 0$, then 1) $\det\|[C_{\alpha\beta}]\| = 0$ and 2) the lines of discontinuity are lines perpendicular to those of the principal axes of the tensor $[C_{\alpha\beta}]$ for which the appropriate eigenvalue $[C_{\alpha\beta}]$ is not zero.

Some estimates of $\Psi_\perp$. Few exact solutions of the problem of the minimum of the functional $\Theta_\perp$ are known, hence, two-sided estimates of $\Psi_\perp$, that bracket the effective one-dimensional energy coefficients acquire special value. We present some of the simplest estimates for inhomogeneous rods.

Upper estimate. We take the trial function y_α in the form

$$y_\alpha = a\zeta_\alpha + (\delta_{\alpha\beta}a_\gamma - {}^1/_2\delta_{\beta\gamma}a_\alpha)(\zeta^\beta\zeta^\gamma - \langle\zeta^\beta\zeta^\gamma\rangle/|S|),$$
$$a, a_\alpha = \text{const}$$

Evaluation of $\Theta_\perp$ as a function of the parameters a and a_α and minimization with respect to a and a_α result in the inequality

$$\begin{aligned}\Psi_\perp \leqslant\ & 2[\langle(\lambda+\mu)\nu^2\rangle - \langle(\lambda+\mu)\nu\rangle^2\langle\lambda+\mu\rangle^{-1} - D_{\alpha\beta}(\langle(\lambda+\mu)\nu\zeta^\alpha\rangle - \langle(\lambda+\mu)\nu\rangle\langle(\lambda+\mu)\zeta^\alpha\rangle\langle\lambda+\mu\rangle^{-1})\times \\ & (\alpha\to\beta)]\,\gamma^2 + 2[\langle(\lambda+\mu)\nu^2\zeta^\alpha\zeta^\beta\rangle - \langle(\lambda+\mu)\nu\zeta^\alpha\rangle\langle(\lambda+\mu)\nu\zeta^\beta\rangle\langle\lambda+\mu\rangle^{-1} - D_{\gamma\delta}(\langle(\lambda+\mu)\nu\zeta^\gamma\zeta^\alpha\rangle - \\ & \langle(\lambda+\mu)\zeta^\gamma\rangle\langle(\lambda+\mu)\nu\zeta^\alpha\rangle\langle\lambda+\mu\rangle^{-1})(\gamma\to\delta,\alpha\to\beta)]\,h^2\Omega_\alpha\Omega_\beta + 4[\langle(\lambda+\mu)\nu^2\zeta^\alpha\rangle - \langle(\lambda+\mu)\nu\rangle\langle(\lambda+\mu)\nu\zeta^\alpha\rangle\langle\lambda+\mu\rangle^{-1} - \\ & D_{\beta\gamma}(\langle(\lambda+\mu)\nu\zeta^\beta\rangle - \langle(\lambda+\mu)\nu\rangle\langle(\lambda+\mu)\zeta^\beta\rangle\langle\lambda+\mu\rangle^{-1})\times(\langle(\lambda+\mu)\nu\zeta^\gamma\zeta^\alpha\rangle - \langle(\lambda+\mu)\zeta^\gamma\rangle\langle(\lambda+\mu)\nu\zeta^\alpha\rangle\langle\lambda+\mu\rangle^{-1})]\,\gamma h\Omega_\alpha\end{aligned} \tag{8.9}$$

Here $D_{\alpha\beta}$ denotes the tensor inverse to the tensor

$$\langle(\lambda+\mu)\zeta^\alpha\zeta^\beta\rangle - \langle(\lambda+\mu)\zeta^\alpha\rangle\langle(\lambda+\mu)\zeta^\beta\rangle\langle\lambda+\mu\rangle^{-1}$$

In particular, we examine a rod with a centrally-symmetric transverse section and even functions λ and μ. Then $\langle(\lambda+\mu)\nu\zeta^\alpha\rangle = 0$ and the estimate (8.9) for the addition of $E_\perp$ to the effective Young's modulus yields

$$E_\perp \leqslant 4\left(\langle(\lambda+\mu)\nu^2\rangle - \langle(\lambda+\mu)\nu\rangle^2\langle\lambda+\mu\rangle^{-1}\right)/|S| \tag{8.10}$$

Let the rod consist of two homogeneous materials whose elastic characteristics are denoted by the subscripts 1 and 2, and the relative areas occupied by α and $1-\alpha$. The inequality (8.10) becomes

$$E_\perp \leqslant 4\alpha(1-\alpha)(\nu_2-\nu_1)^2\left[\alpha(\lambda_2+\mu_2)^{-1}+(1-\alpha)(\lambda_1+\mu_1)^{-1}\right]^{-1} \tag{8.11}$$

For comparison we present the exact value of $E_\perp$ found in /17/ for a circular rod for which the Lamé parameters have the values λ_1, μ_1 for $|\xi^1|^2+|\xi^2|^2 \leqslant R_1^2$ and λ_2, μ_2 for $R_1 \leqslant |\xi^1|^2 + |\xi^2|^2 \leqslant R_2$

$$E_\perp = 4\alpha(1-\alpha)(\nu_2-\nu_1)^2\left[\alpha(\lambda_2+\mu_2)^{-1}+(1-\alpha)(\lambda_1+\mu_1)^{-1}+\mu_2^{-1}\right]^{-1} \tag{8.12}$$

Comparing (8.11) and (8.12) shows that the estimate is asymptotically exact for $\mu_1/\mu_2 \to 0$ and its error increases together with μ_1/μ_2.

Lower estimate. In contrast to the upper estimate, a lower estimate is not constructed successfully by using the dual variational problem (7.1) for any cross-section since the function χ should satisfy the boundary conditions (7.2) on the section boundary. To illustrate application of the dual problem, we obtain the lower estimate of $E_\perp$ for an isotropic circular rod with parameters λ, μ varying over the section. We take the function χ in the form $\chi = a(\zeta_\alpha\zeta^\alpha - 1/4)^2$. Here conditions (7.2) are satisfied. Substitution into (7.1) and maximization with respect to the parameter a yield

$$E_\perp \geqslant 4\langle\nu(\zeta_\alpha\zeta^\alpha - 1/8)\rangle^2\left[1/4\langle\mu^{-1}(\zeta_\alpha\zeta^\alpha)^2\rangle + \langle(\lambda+\mu)^{-1}(\zeta_\alpha\zeta^\alpha - 1/8)^2\rangle\right]^{-1}|S|^{-1}$$

Let us note that $\langle\nu(\zeta^\alpha\zeta_\alpha - 1/8)\rangle = \langle(\nu - \langle\nu\rangle|S|^{-1})\zeta_\alpha\zeta^\alpha\rangle$ and the right side of the lower estimate vanishes, as it should, for $\nu = \text{const}$.

9. Energy of a rod from physically linear material. The rod deformation is determined completely by γ, Ω_α and Ω, hence, in a first approximation the principal terms in γ, Ω_α and Ω and the principal cross terms must be retained in the expression for the energy density Φ. The principal terms in γ, Ω_α, and Ω are contained in (1.6). As is seen from the results in Sects. 7 and 8, the cross terms in (1.6) can vanish and the principal cross terms turn out to be among the energy components of the order of $\mu h^2(\varepsilon^3 + h_*\varepsilon^2 + h_{**}\varepsilon^2)$. We first consider the energy Φ (1.6) of the order of $\mu h^2\varepsilon^2$, and we then evaluate the cross terms.

Homogeneous anisotropic rods. From (1.6) and (7.3) we obtain

$$2\Phi = E|S|\gamma^2 + EI^{\alpha\beta}\Omega_\alpha\Omega_\beta + C(\Omega - 1/2 e^{\mu\nu}C_\mu\Omega_\nu)^2 \tag{9.1}$$

If the elastic properties of the rod are symmetric with respect to a plane perpendicular to the axis, then (9.1) goes over into the classical expression

$$2\Phi = E|S|\gamma^2 + EI^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2 \tag{9.2}$$

In the opposite case, the component $-Ce^{\mu\nu}C_\mu\Omega_\nu\Omega$ that describes the cross interaction between the torsion and the bending enters the expression for the energy, and the effective bending stiffnesses are determined by not only the tensor of the moments of inertia of the section ($EI^{\alpha\beta}$ is as in the classical case) but also by an addition associated with the torsional stiffness. The bending stiffnesses have the form $EI^{\alpha\beta} + 1/4\, Ce^{\mu\alpha}e^{\nu\beta}C_\mu C_\nu$. We note that the cross interaction and the increase in bending stiffness occur only for bending in the plane perpendicular to the vector C_α.

We shall later examine rods whose elastic properties are symmetric relative to a plane perpendicular to the axis.

Inhomogeneous rods. According to (1.6) and (8.1)

$$2\Phi = [(\langle E\rangle + E_\perp |S|)\gamma^2 + 2(\langle E\xi^\alpha\rangle + E_\perp^\alpha |S|)\gamma\Omega_\alpha + (\langle E\xi^\alpha\xi^\beta\rangle + E_\perp^{\alpha\beta}|S|)\Omega_\alpha\Omega_\beta] + C\Omega^2 \tag{9.3}$$

An interesting observation made for the isotropic case by N.I. Muskhelishvili /17/, is associated with this equality and (8.2). Let there be two rods with identical values of $\langle E\rangle$, $\langle E\xi^\alpha\rangle$ and $\langle E\xi^\alpha\xi^\beta\rangle$ and different values of the Poisson's ratios $C_{\alpha\beta}$ where the Poisson's ratios are constant for one rod. Then the bending stiffness and the Young's modulus are greater in the rod whose Posson's ratios are variable (if the latter do not here satisfy the criterions of $\Psi_\perp$ degeneration (8.3) and (8.4); N.I. Muskhelishvili considered piecewise-homogeneous isotropic rods for which there can be no degeneration of $\Psi_\perp$ according to Corollary 2 of Sect.8).

For rods with a centrally symmetric cross-section and even elastic properties $\langle E\xi^\alpha\rangle = E_\perp{}^\alpha = 0$ and the cross effect between the tension and bending vanishes.

Cross terms. We limit ourselves to an analysis of homogeneous anisotropic rods with centrally-symmetric section and we take $h_* \ll \varepsilon$ so that terms on the order of $h_*\varepsilon^2$ can be neglected as compared with terms on the order of ε^3; ; this latter assumption is valid, in particular, for straight rods in the undeformed state . Cross terms on the order of $h_*\varepsilon^2$ were constructed in /3,18/.

It can be confirmed that only the following components of the expression for the strain

$$2\varepsilon_{\alpha 3} = y_{|\alpha} + h\Omega(1 - C_\mu^\mu\gamma)e_{\sigma\alpha}\zeta^\sigma - h\frac{d\gamma}{ds}C_{\alpha\sigma}\zeta^\sigma \tag{9.4}$$

$$\varepsilon_{33} = \gamma + [\delta_{\alpha\beta} + (\delta_{\alpha\beta} - C_{\alpha\beta})\gamma]h\Omega^\alpha\zeta^\beta + {}^1\!/_2 h^2(\Omega_\alpha\zeta^\alpha)^2 + {}^1\!/_2 h^2\Omega^2\zeta_\alpha\zeta^\alpha$$

will yield contributions to the energy because of the central symmetry of the cross-section and the evenness properties of y and y_α .

The expressions (9.4) contain corrections of the order of ε and h_{**} as compared to one.

In conformity with the general scheme of the variational-asymptotic method /2/, y and y_α must be represented in the form $y = y_0 + y'$, $y_\alpha = y_{0\alpha} + y_\alpha{}'$ to construct all the corrections of order ε and h_{**}, where y_0, $y_{0\alpha}$ are the minimizing functionals of the functional Θ, the principal terms in y', $y_\alpha{}'$ and the principal cross terms must be retained in the expression for the energy density of a three-dimensional body, and the obtained functional must then be minimized with respect to y', $y_\alpha{}'$. It can be confirmed that the quantity $y_\alpha{}'$will hence turn out to be of the order of ε^2 and can be discarded since no cross terms between $y_{0\alpha}$ and $y_\alpha{}'$will enter the expression for the energy because of the Euler equations for the functional $\Theta_\perp$, and the determination of the energy with corrections on the order of ε and h_{**} taken into account is equivalent to replacing the torsion problem by a "refined" torsion problem

$$^1\!/_2 G^{\alpha\beta}\langle(y_{|\alpha} + h\Omega(1 - C_\mu^\mu\gamma)e_{\sigma\alpha}\zeta^\sigma - C_{\alpha\sigma}\zeta^\sigma h\,d\gamma/ds)(\alpha\to\beta)\rangle \to \inf_y \tag{9.5}$$

The substitution $y \to z$

$$y = {}^1\!/_2 C_{\alpha\beta}(\zeta^\alpha\zeta^\beta - \langle\zeta^\alpha\zeta^\beta\rangle / |S|)h\,d\gamma/ds + h\Omega(1 - C_\mu^\mu\gamma)z$$

reduces it to a torsion problem. Hence, the minimum of the functional (9.5) equals $^1\!/_2 C\Omega^2(1 - C_\mu^\mu\gamma)^2$. Taking account of the expression for ε_{33} in (9.4), we find for the rod energy density

$$\Phi = {}^1\!/_2(E|S|\gamma^2 + EI^{\alpha\beta}\Omega_\alpha\Omega_\beta + C\Omega^2) + B\gamma\Omega^2 + B_\beta^\alpha I_{\alpha\sigma}\gamma\Omega^\beta\Omega^\sigma \tag{9.6}$$

The constrants B and $B_\beta{}^\alpha$ characterize the cross effects between the tension and the torsion, the tension and the bending, and are given by the formulas

$$B = {}^1\!/_2 EI_\alpha^\alpha - C_\alpha^\alpha C, \quad B_\beta^\alpha = E({}^3\!/_2\delta_\beta^\alpha - C_\beta^\alpha)$$

The error in (9.6) is $O(\mu h^2(h_{**}\varepsilon^3 + \varepsilon^4))$. The expression (9.6) does not contain a cross term between the torsion and the bending since it is of the order $\mu h^2 h_{**}\varepsilon^3$.

In the isotropic case the tensor $B_\beta{}^\alpha$ is spherical and positive definite. The constant B and the inequality $(I^{-1\alpha}_{\ \alpha})^{-1} \leqslant {}^1\!/_4 I_\alpha^\alpha$ are positive, as follows from the estimate (7.11). This means that the twisting of the rod causes its shortening, if the ends are free and a tensile force is induced if the ends are fixed. However, this will hold only for isotropic rods. In the anisotropic case, the constant B can be positive because of the arbitrariness of G, E and ν , and can also be negative.

This can be explained as follows. Cross interaction between the tension and torsion is due to two geometrically nonlinear effects. Elongation of the fibers parallel to the axis occurs during twisting. It generates an increase in the rod energy (the first term in the coefficient B). On the other hand, longitudinal elongation because of the Poisson effect is accompanied by transverse deformation, which causes (a geometrically nonlinear effect!) an additional shear strain that diminishes the shear energy (the second component in the coefficient B for $C_\mu^\mu > 0$). In the isotropic case the first factor predominates and the energy increases, while the inverse can occur for anisotropic rods.

We present the value of the constant B for a circular isotropic rod of radius R: $B = {}^1/_4\pi R^4 E(1-\nu)/(1+\nu)$.

One-dimensional functional. The equilibrium position of an elastic rod can be sought, in a first approximation, from the condition of stationarity of the functional (1.5) in the set of functions $r^i(\xi)$ and $\tau_\alpha^i(\xi)$ that satisfy the constraints (1.1) and the boundary conditions on the endfaces (3.5). In conformity with (2.4), (3.2), (5.2) and the estimate $y^i = O(\varepsilon)$, the work of the external forces in (1.5) has the form

$$L = \int_0^{|\Gamma_0|} (Q_i r^i + Q_i^\alpha \tau_\alpha^i)\, d\xi, \quad Q_i = h \int_{\partial S} P_i\, d\sigma + F_i |S|,$$

$$Q_i^\alpha = h^2 \int_{\partial S} P_i \zeta^\alpha\, d\sigma$$

10. Energy of a rod from Murnaghan and Mooney material.

It is natural to retain contributions of order ε^3 even in the energy of a three-dimensional body in taking account of the cross terms of order ε^3 in the one-dimensional energy. By using the variational asymptotic method it can be found that the appropriate changes in Φ reduce to adding values of terms on the order of ε^3 of the energy of a three-dimensional body, calculated in the strains (6.1). An expression of the form (9.6) with coefficients B and $B_\beta{}^\alpha$ given by the formulas

$$B = {}^1/_2\, EI_\mu{}^\mu + C\,({}^1/_2\,(1-2\nu)\, m\mu^{-1} + {}^1/_2 \nu n \mu^{-1} - 2\nu)$$

$$B_\beta{}^\alpha = \delta_\beta{}^\alpha\,[({}^3/_2 - \nu)\,E + (1-2\nu)^3\,(l-m) + 3\,(1-2\nu) \times (1+2\nu^2)\, m + 3\nu^2 n]$$

for a Murnaghan material (l, m, n are the Murnaghan constants /19/) is hence obtained for the energy.

The corresponding expressions for the Mooney material have the form (c is the Mooney constant /19/)

$$B = {}^1/_2\, EI_\mu{}^\mu - {}^1/_4\,(7+c)C, \quad B_\beta{}^\alpha = {}^1/_2\, E\,(1+c)\, C.$$

REFERENCES

1. BERDICHEVSKII V.L., On equations of the theory of anisotropic inhomogeneous rods, Dokl. Akad. Nauk SSSR, Vol.228, No.3, 559, 1976.
2. BERDICHEVSKII V.L., Variational-asymptotic method of constructing a theory of shells, PMM, Vol.43, No.4, 1979.
3. BERDICHEVSKII V.L., and STAROSEL'SKII L.A., On the theory of naturally twisted rods, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.6, 1979.
4. PONIATOVSKII V.V., Application of an asymptotic integration method to the equilibrium problem of a thin bar loaded arbitrarily on the side surface, Inzh. Zh., Mekhan., Tverd. Tela, No.5, 1968.
5. RIGOLOT A., Theorie asymptotique des milieux curvilignes et equilibre élastique d'un cylindre infiniment élancé, C.R. Acad. Sci., Vol.272, No.11, 1971.
6. PONIATOVSKII V.V., Asymptotic theory of the bending of curved bar. IN: Investigations on Elasticity and Plasticity, No.9, Lengingrad State University Press. 1973.
7. ELISEEV V.V., Application of an asymptotic method in the equilibrium problem of a curvilinear rod, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, NO.3, 1977.
8. LADEVÉZE J., LADEVÉZE P., MANTION M., PÉCASTAINGS, F, and PELLE J.-P., Sur les fondements de la théorie lineáre des poutres élastiques. I., J. Mec., Vol.18, No.1, 1979.
9. SEDOV L.I., Mechanics of Continuous Medium, Vol.1, Wolters-Noordhoff, 1971.

10. BERDICHEVSKII, V.L., On a variational principle, Dokl. Akad. Nauk SSSR, Vol.215, No.6, 1974.
11. VOIGT W., Lehrbuch der Kristallphysik, TEUBNER, Leipzig—Berlin, 1928.
12. ILIUKHIN A.A., Spatial Problems of the Nonlinear Theory of Elastic Rods. NAUKOVA DUMKA, Kiev, 1979.
13. LEKHNITSKII S.G., Torsion of Anisotropic and Inhomogeneous Rods, NAUKA, Moscow, 1971.
14. NIKOLAI E.L., Research in Mechanics. GOSTEKHTEORIZDAT, Moscow, 1955.
15. LOVE A., Mathematical Theory of Elasticity, GLAV. RED. OBSHCHETEKHN. Lit-RY I NOMOGR.,1935.
16. ARUTIUNIAN N.Kh., and ABRAMIAN B.L., Torsion of Elastic Bodies. FIZMATGIZ, Moscow, 1963.
17. MUSKHELISHVILI N.I., Some Fundamental Problems of the Mathematical Theory of Elasticity. Groningen, Noordhoff, 1953.
18. STAROSEL'SKII L.A., On equations describing the vibrations of curvilinear elastic rods, Dokl. Akad. Nauk SSSR, Vol.247. No.1, 1979.
19. LUR'E, A.I., Theory of Elasticity, NAUKA, Moscow, 1970.

Translated by M.D.F.